Thin film-based Gires–Tournois Resonator (GTR) as quasi-optical analogue of the Thomas rotation angle effect in special relativity

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ABSTRACT
We present an unconventional approach to generate an optical analogue of the Thomas rotation angle found in Special Relativity (SR) by using a thin-film–based Gires–Tournois Resonator (GTR). At first glance, the phase response of the GTR seems to fail as an optical analogue for the Thomas rotation angle in the phase domain but we demonstrate that an analogue can be successfully constructed in the intensity domain by combining the GTR with a Michelson interferometer (MI), thus forming an interferometer known as the Michelson–Gires–Tournois Interferometer (MGTI). Thomas rotation angle is a spatial rotation of the reference frame due to the Einstein velocity addition (EVA) law of two successive relativistic velocities travelling in non-collinear directions, whereas, the GTR is a thin-film based asymmetric Fabry–Perot resonator (FPR) with a partially reflecting front mirror, and a 100% rear-reflecting mirror. We investigate in detail, both analytically and numerically, the Thomas rotation angle’s full behaviour under various parameter conditions. This approach offers the following advantages: (i) it leads to a simpler overall configuration, (ii) it measures the Thomas rotation angle directly requiring only a single optical parameter instead of multiple parameters compared with other methods, and (iii) it requires no sophisticated multilayer thin film design.

1. Introduction

In recent years, the close mathematical similarities between Special Relativity (SR) and Multilayer Optics (MO) (also known as stratified layered media, parallel resonators, layer optics, or thin films) have attracted significant interest in the scientific community [1–16]. These similarities or analogies enrich both fields with their useful physical, geometric, and mathematical interpretations. Thus, MO becomes a potentially powerful and readily accessible platform to mimic SR phenomena using only simple measurements in the optical laboratory.

Vigoureux [1–3] proposed a formal analogy between (i) the composition law of reflected complex field amplitudes in MO, and (ii) the composition law of relativistic velocities (or the Einstein velocity addition, EVA law) in SR. Then, Monzon and Sánchez-Soto [4–6] extended this analogy further to (i) both the reflection and transmission coefficients of the complex field amplitudes in MO, and (ii) their corresponding parameters of the Lorentz transformation (LT). Moreover, Vigoureux and co-workers [7–9] also applied their mathematical analogy to Polarization Optics with the analogy between (i) the composition law for polarizers, and (ii) the familiar composition law for non-collinear velocities. Other researchers have also developed further the link between Polarization Optics and Lorentz transformation in more detail [10–12].

These three different realizations of the MO-vs-SR analogy are illustrated and briefly summarized in Fig. 1. The first realization (here referred to as Type 1) focuses solely on the reflected complex electric field, E R from the multilayer optics (depicted here as a Fabry–Perot Resonator, FPR). The FPR consists of two parallel partially reflecting mirrors separated by a certain defined distance d [13,14]. The second realization (referred to as Type 2) includes the transmitted electric field E T of the FPR, as well as the E R component of the complex field signal, which are formalized in the language of the Lorentz transformation [4–6]. On the other hand, the third realization (referred to as Type 3) covers the polarization state of the transmitted optical signal when it interacts with the two series of polarizers. The polarization state can be related to Einstein velocity addition in SR [7–9].

Lastly, our proposed scheme is depicted as Type 4 to highlight its differences compared with Type 1 and Type 2 by using a Gires–Tournois Resonator (GTR) [17,18] instead of FPR. This new approach can also be considered as a unique subset of Type 1, since the GTR is a special case of the FPR. This will be discussed in detail in Section 3.
From a theoretical viewpoint, the first three realizations establish firmly the fundamental close links between SR and MO. As direct results of this analogy, these three realizations could potentially open the path to mimic many SR phenomena using only simple measurements that can be performed readily in the optical laboratory.

Unfortunately, the experimental side of the goal has met very limited successes [15,16] because (i) only few research investigations have focused on this issue and (ii) most of these SR phenomena are not easily translatable in a simple experimental setup with a one-to-one correspondence between the intended SR phenomena and target optical parameter. One such experimentally challenging SR phenomenon is the Thomas rotation angle or Wigner angle Effect [19,20]. It is the spatial (or coordinate axis) rotation of the observed reference frame due to the Einstein velocity addition (EVA) law of two successive relativistic velocities, \( \vec{V}_1 \) and \( \vec{V}_2 \), travelling in a non-collinear direction with relative angle \( \theta \).

Despite the numerous published papers on SR and MO for the last three decades, to the best of our knowledge, we are aware of only one thin film-based experimental proposal to mimic the Thomas Effect. It was based on the theoretical finding [4] from Type 2 that the Thomas rotation angle \( \varepsilon \) is directly associated with the phase of the transmitted complex field signal from the “two specially designed compound multilayers” put together back-to-back. Unfortunately, these compound multilayers have specifications that are demanding to fabricate, namely: (i) perfectly-matched, (ii) lossless, and (iii) zero-total-reflection (or perfect anti-reflection coating). Furthermore, there was little to no discussion on how to implement in an actual experimental setup the proposed Thomas rotation angle analogue as a function of the generating angle \( \theta \).

In comparison with the Type 2 approach, the approaches based on Type 1 and Type 3 do not lean themselves easily to the generation of an optical analogue of the Thomas rotation angle that can be measured readily and directly in a straightforward manner [21]. Physically, the reason for this difficulty is that the reflected phase of the complex amplitude in FPR is not mathematically equivalent to the Thomas rotation angle in the same manner, as in the case of the phase response of the transmitted signal from the compound multilayer identified in [4].

In this paper, we present, for the first time to the best of our knowledge, an unconventional approach to generate a quasi-optical analogue of the Thomas rotation angle using a Gires–Tournois resonator (GTR). Structurally speaking, the GTR is a FPR with a partially reflecting front surface but with a 100% reflecting back surface [17,18]. This approach has two novelties. First, we started with an imperfect analogue of the Thomas rotation angle by using a GTR. This action, at first glance, is contrary to common practice in the literature since mathematically speaking, the equation of the phase response of the GTR bears only a little similarity to that of the equation describing the Thomas rotation angle. Clearly, the GTR is not an obvious candidate for any Thomas rotation angle analogue and has never been considered as an important vehicle to study SR in the same way as FPR. However, in conjunction with a correction scheme, we demonstrate both analytically and numerically, that we can successfully construct the correct Thomas rotation angle analogue.

The second novelty is the simple and yet effective all-optical-correction scheme we employed to correctly “shape” the phase response of the GTR in order to generate a correct Thomas rotation angle analogue. This is realized by combining the GTR with a Michelson interferometer (MI) to form the Michelson–Gires–Tournois Interferometer (MGTI) [22,23]. The MGTI is a typical MI in which one of the reflecting mirrors is replaced by a GTR. MGTI is a mature technology and is frequently used in many applications in optics and photonics such as bandpass filter and narrowband filter [22], optical interleaver [24–26], modulator [27,28] to name a few. In our particular case, the MGTI performs two important tasks namely:

1. the correction of the phase response of the GTR by choosing the appropriate path-length difference, \( \Delta L \) in the unbalanced MGTI, and
2. the conversion of the Thomas angle-encoded information found in the phase response of the output signal into intensity for direct measurement.

Overall, this unconventional approach leads to the following benefits. First, it leads to a straightforward and simple thin-film-based optical analogue of the Thomas rotation angle with a simpler optical configuration. In particular, it is able to associate the Thomas rotation angle \( \varepsilon \) to only a single optical parameter that leads to a direct and readily measurable quantity. Second, it enables the easy implementation of the Thomas angle analogue as a function of the generating angle \( \theta \).
angle θ. Third, it requires no sophisticated multilayer thin film design and coating compared with the Type 2 approach mentioned earlier.

This paper is organized as follows. First, in Section 2, we briefly review Einstein’s velocity addition rule in the complex plane to give the readers enough mathematical and SR background. In Section 3, we briefly summarize the challenges in developing an analogue for Thomas angle using Type 1 structure. In Section 4, we discuss the formulation of the GTR, and establish the imperfect analogue between the Thomas angle and the original phase response of the GTR. In Section 5, we address the problem of correcting this imperfect analogue by combining the GTR with the unbalanced Michelson Interferometer. Lastly, in Section 6, we give our conclusion.

2. Einstein velocity addition rule and the Thomas rotation angle

The typical three-coordinate model used to discuss Einstein velocity addition (EVA) [29] is shown in Fig. 2a. It consists of (i) the initial rest frame Σ, (ii) the second frame Σ′ moving with a velocity V1 along the Y-axis in a parallel direction relative to the initial rest frame Σ, and (iii) a third frame Σ″ travelling with velocity V2 in parallel direction with respect to the frame Σ′. However, we note that the third coordinate frame Σ″ is travelling in a non-collinear direction relative to the frame Σ and makes an angle θ1 between the velocities V1 and V2. In SR literature, the angle θ1 is referred to as the generating angle. As depicted in Fig. 2b, the resultant addition of two velocities leads to the expected combined Lorentz boost (Ṽ1 ⊕ Ṽ2) output located at frame Σ‴. Furthermore, the resultant addition of the two velocities also generated a counter-intuitive spatial rotation of frame Σ‴ (circled dashed, red coloured axis) as perceived by an observer at the initial rest frame Σ as shown in Fig. 2c. This phenomenon is a physical manifestation of the Thomas rotation angle ε. When the two velocities V1 and V2 travel collinearly, then the Thomas rotation angle ε is zero.

We define the sign of the directional angle θ1 as follows. We assumed that the two non-collinear velocities V1 and V2 make an angle, θ2, travelling towards some arbitrary positive-designated coordinate system. Here, we take the angle θ2 to be positive-valued when it is directed counter-clockwise, as shown in Fig. 2b. Subsequently, it is negative-valued when it is directed in a clockwise direction.

2.1. Einstein’s addition rule extended to the complex plane

Next, we employ Vigoureux’s formulation of the EVA [12,30,31] due to its simplicity instead of the traditional treatment using the Lorentz transformation. This complex plane formulation has been covered in large detail and has been applied to various applications [6–9, 32–40]. In this section, we briefly provide only the essential features and encourage readers to consult the references for more detailed information. Vigoureux [21,41] defined the EVA for the non-collinear coplanar velocities as,

\[ \overline{\mathbf{W}_{1\mathbb{G}2}} = \overline{V}_1 \oplus \overline{V}_2 = \overline{V}_1 + \frac{\overline{V}_2}{1 + \overline{V}_2 \overline{V}_1} \]  

where the respective complex velocities are defined as

\[ \overline{V}_1 = V_1 e^{i \beta_1} = \tanh \frac{V_1}{c} e^{i \beta_1}, \quad \overline{V}_2 = V_2 e^{i \beta_2} = \tanh \frac{V_2}{c} e^{i \beta_2} \]

with the symbol \( \oplus \) in Eq. (1) denoting the Einstein addition operation while the asterisk * denotes the complex conjugate of \( \overline{V}_1 \). Note that the hyperbolic angles, \( \beta_1 \) and \( \beta_2 \), are the rapidity values associated with the standard speeds \( v_1 \) and \( v_2 \) defined as

\[ \beta_1 = \tanh^{-1} \left( \frac{V_1}{c} \right), \quad \beta_2 = \tanh^{-1} \left( \frac{V_2}{c} \right) \]

with \( c \) being the speed of light. Furthermore, the angles \( \theta_1 \) and \( \theta_2 \) in Eq. (2) indicate the directions of the complex velocities \( \overline{V}_1 \) and \( \overline{V}_2 \), respectively. We note that the magnitudes \( |V_1| \) and \( |V_2| \) in Eq. (2) of the complex velocities are limited to the closed interval \([−1, 1]\).

From Eq. (1), the resultant velocity, \( \overline{W}_{1\mathbb{G}2} \), is expressed as

\[ \overline{W}_{1\mathbb{G}2} = \overline{W}_{1\mathbb{G}1} e^{i \theta_1} = \overline{V}_1 e^{i \beta_1} + \overline{V}_2 e^{i \beta_2} = \frac{V_1 + V_2 e^{i(\theta_2-\theta_1)}}{1 + V_1 V_2 e^{i(\theta_2-\theta_1)}} e^{i \theta_1} \]

where the term \( \overline{W}_{1\mathbb{G}1} \) is the overall modulus, and \( \overline{W}_{1\mathbb{G}2} \) is the directional angle of the resultant velocities, \( \overline{W}_{1\mathbb{G}2} \). If we interchange the sequence of the velocities (\( V_2 \) to \( V_1 \), instead of \( V_1 \) to \( V_2 \)), Eq. (4) becomes

\[ \overline{W}_{2\mathbb{G}1} = \overline{W}_{2\mathbb{G}0} e^{i \theta_2} = \frac{V_2 e^{i \beta_2} + V_1 e^{i \beta_1}}{1 + V_1 V_2 e^{i(\theta_2-\theta_1)}} e^{i \theta_2} \]

where \( \overline{W}_{2\mathbb{G}0} \) and \( \overline{W}_{2\mathbb{G}1} \) are their corresponding modulus and directional angle, respectively. It must be noted that the sign in the exponential term in the denominator in Eq. (5) is negative compared with the denominator sign in Eq. (4) due to the conjugate operation performed with velocity \( V_2 \). It can be easily shown that the magnitudes of \( \overline{W}_{1\mathbb{G}2} \) and \( \overline{W}_{2\mathbb{G}1} \) are equal.

2.2. Calculation of the Thomas rotation angle

The Thomas rotation angle \( \epsilon \) is defined as the net rotation after its trajectory returns back to its initial frame [1–7,21,39]. Mathematically speaking, this translates to taking the ratio between the two resultant velocities, \( \overline{W}_{1\mathbb{G}2} \) and \( \overline{W}_{2\mathbb{G}0} \). Note that the magnitude, \( | \overline{W}_{1\mathbb{G}2} | \), is equal to magnitude of \( | \overline{W}_{2\mathbb{G}1} | \). Thus, their directional angles \( \overline{W}_{1\mathbb{G}2} \) and \( \overline{W}_{2\mathbb{G}1} \) in Eqs. (4) and (5), respectively, determine the effective Thomas rotation angle \( \epsilon(\theta, V_1, V_2) \) [21,41], which is derived as

\[ \epsilon(\theta, V_1, V_2) = \epsilon(\theta, V_1, V_2) = \frac{1 + V_1 V_2 e^{i \theta_2}}{1 + V_1 V_2 e^{-i \theta_2}} \]

or as [9]

\[ \epsilon(\theta, V_1, V_2) = \epsilon(\theta, V_1, V_2) = \frac{1 + V_1 V_2 e^{i \theta_2}}{1 + V_1 V_2 e^{-i \theta_2}} \]

where \( \theta = \theta_2 - \theta_1 \). By taking the velocity \( \overline{V}_2 \) to be travelling along an arbitrary positive Y-axis direction, we can set the term \( \theta_1 \) to be equal to zero so that the term \( (\theta_2 - \theta_1) \) is simply \( \theta \), as shown in Fig. 1.

The only difference between Eqs. (6a) and (6b) is the sign of the Thomas angle \( \epsilon \) which is directly related to the choice of the frame of reference or mathematically equivalent to which denominator is used in Eq. (6). As noted in [7,42–46], there is a lack of agreement among researchers on the “proper” sign convention for the Thomas angle. This is due to some confusing definitions of the frame of reference or lack of clear statement about the assumed sign convention employed.

Here we use Eq. (6b). Then, the Thomas rotation angle \( \epsilon(\theta, V_1, V_2) \) is found to be

\[ \epsilon(\theta, V_1, V_2) = - \tan^{-1} \frac{(2 V_1 V_2 \sin \theta)(1 + V_1 V_2 \cos \theta)}{(1 + V_1 V_2 \cos \theta)^2 - (V_1 V_2 \sin \theta)^2} \]

We can simplify further Eq. (7a) by dividing both the numerator and denominator inside the tan⁻¹ term with \( (1 + V_1 V_2 \cos \theta)^2 \) so that Eq. (7a) becomes

\[ \epsilon(\theta, V_1, V_2) = - \tan^{-1} \frac{2 V_1 V_2 \sin \theta}{1 + V_1 V_2 \cos \theta} \]

and by using the identity \( \tan[2Z] = 2 \tan[Z]/(1 - \tan[Z]^2) \), we can express the final result as

\[ \epsilon(\theta, V_1, V_2) = - 2 \tan^{-1} \frac{V_1 V_2 \sin \theta}{1 + V_1 V_2 \cos \theta} \]

The Thomas rotation angle as expressed by Eq. (8) is a very compact expression that provides immediate physical insights compared with Eq. (7a). It is also one of the key aspects of this paper. Eq. (8) is instrumental in identifying and developing the quasi-optical analogue.
we note that when the value of $V$ is negative (left side) and positive (right side) relative to the angle $\theta$ considered as two plots for the cases when the generating angle $\theta$ is negative and positive, respectively.

The typical illustration of the three-coordinate frame (Fig. 2) to discuss Einstein's velocity addition (a) showing the resultant velocity ($V_1 \oplus V_2$) (b) and the Thomas rotation angle $\varepsilon$ (c).

Fig. 3. Thomas angle $\varepsilon$ as a function of the generating angle, $\theta$ for various values of $V_1V_2$. The insets are the corresponding 3-coordinate frame to illustrate the velocity addition $V_1$ and $V_2$ with their respective Thomas rotation angles $\varepsilon$ when the generating angle $\theta$ is negative and positive, respectively.

3. Challenges in generating Thomas angle analogue in Type 1 and Type 3

As pointed out by Vigoureux [21,41], the analogy between (i) the composition law of reflected complex field amplitudes in MO and (ii) the composition law of relativistic velocities gives three important technical results namely: (i) the magnitudes of $|V_1 \oplus V_2|$ and $|V_2 \oplus V_1|$, (ii) the directional angles, $\Psi_{1g2}$ and $\Psi_{2bg1}$, and (iii) the orientation of the observed frame $\Sigma''$ relative to the initial rest frame $\Sigma$. The first two quantities (i.e. magnitudes and directional angles) have direct one-to-one correspondences with the optical parameters in FPR that can be measured readily in the optical laboratory.

However, the third quantity does not seem to be directly accessible in the same way with the first two quantities [21,41]! This poses a challenge for generating an analogue of the Thomas rotation angle $\varepsilon$ based on Type 1 structure. We note that the orientation of the observed reference frame $\Sigma''$ (as perceived by observer in frame $\Sigma$) is directly related and characterized by the Thomas rotation angle $\varepsilon$ [19–21].

By definition, the Thomas rotation angle involves the ratio of two expressions involving the returning ($\overrightarrow{W}_{2bg1}$) and the forward ($\overrightarrow{W}_{1g2}$) Einstein velocity additions as expressed by Eq. (6). Following the analogy for Type 1 structure, this translates to the ratio of the two reflected phases of the complex amplitudes from FPR. This presents two experimental difficulties for multilayer optics Type 1 (FPR) and Type 3 structures. First, the reflected phase of the complex amplitude in Type 1 (FPR) structure is far from being a mathematical equivalent of the Thomas rotation angle in the same manner, as in the case of the phase response of the transmitted signal of the compound multilayer mentioned earlier for Type 2 structure [4]. Second, the ratio requirement makes it experimentally difficult to find a single optical parameter in multilayer optics that can be associated directly with the Thomas rotation angle.

Although it is reasonable to construct an optical system to measure the two phase responses involved in this ratio indirectly and separately, the resultant optical configuration becomes complicated and would involve additional signal processing [15,16].

These same measurement challenges mentioned above are also found with Type 3 structure wherein one works with the polarization states of the two polarizers [7–9] instead of the two reflected complex amplitudes found in Type 1 (FPR) structure.

In the next section, we will show that by starting with a GTR configuration instead of the FPR and using Vigoureux's analogy between the composition law of reflected complex field amplitudes and EVA, we get an imperfect analogue of the Thomas angle (in the phase domain) but
this can be corrected successfully (in the intensity domain) to arrive at a directly measurable single-parameter quantity for the Thomas rotation angle. We note that this quantity involves only one optical parameter unlike the two-parameter measurements required by Eq. (6) for Type 1 (FPR) case.

4. Gires–Tournois Resonator (GTR)

In this section, we briefly review the principle and configuration of the GTR. Then, we discuss the similarities and differences between the equations for the Thomas rotation angle in SR and the phase response of the GTR. This will show us that we can theoretically construct a partial analogy with the Thomas rotation angle.

4.1. Configuration of the GTR

Fig. 4a illustrates a configuration of the GTR. It is basically an asymmetric FPR with (i) a partially reflecting mirror, M1 characterized by a reflectance coefficient \( r_1 \) and (ii) a 100% back reflecting mirror, M2 with reflectance coefficient \( r_2 = 1 \) [17,18]. These two mirrors (M1 and M2) are separated by a distance \( d \) and the material between the two mirrors is denoted by a refractive index \( \eta = 1 \) for simplicity as in Fig. 4a.

In optics, the thin-film based GTR is a mature technology that has had many applications such as chromatic dispersion compensation for optical telecommunication [47,48], ultrawide pulse compression [49,50], multiplexers [51], phase modulator and wavefront transformer [52] to name a few. Due to these important applications, other metrology-related technology associated with GTR have also been developed that would be useful for our purpose.

Fig. 4b depicts the top-view of the GTR showing the multiple reflections of the signal when the input signal beam \( E_{in} \) is incident at an angle \( \phi \). Assuming a lossless GTR and with an input signal, \( E_{in} \) incident normal onto the GTR (as in the case in Fig. 4a), then the output complex electric field signal, \( E_{out} \) reflected from the GTR is derived as [17,18,41,47–49]

\[
\frac{E_{out}}{E_{in}} = E_{GTR} e^{i\Phi_{GTR}} = \frac{r_1 + e^{i\psi}}{1 + r_1 e^{i\psi}}
\]  

(9)

where the term \( \psi = 2k_0d \) is the single-pass phase shift of the GTR, \( d \) is the resonator length, \( \eta \) is the refractive index of the resonator, \( k = 2\pi / \lambda \), and \( \lambda \) is the wavelength of the input signal.

Here, we employ the positive convention \( e^{i\psi} \) instead of the traditional negative \( e^{-i\psi} \) expression which is used typically to represent a travelling wave signal in the optics and photonics fields. The rationale for this choice of convention is to be consistent with the convention in Section 2 that we used to represent a travelling object directed to some positive coordinate system having a directional angle \( (\psi_{102} \text{ or } \psi_{201}) \) when discussing SR.

By factoring out the term \( e^{i\psi} \) in Eq. (9), we get

\[
\frac{E_{out}}{E_{in}} = E_{GTR} e^{i\Phi_{GTR}} = \frac{1 + r_1 e^{-i\psi}}{1 + r_1 e^{i\psi}} e^{i\psi}.
\]  

(10)

Because the back mirror is fully reflecting and assuming lossless material in GTR, all of the energy of the input signal (regardless of the frequencies or wavelengths) are reflected back eventually toward the direction of the light source after circulating many times inside the resonator. Thus, the ratio between \( E_{out} \) and \( E_{in} \) should be equal to unity. The duration of the signal circulating inside the resonator (also known as the time or group delay) will depend on the parameter \( r_1 \) and the particular incident wavelength.

On the other hand, the corresponding phase response, \( \Phi_{GTR} \) of the GTR can easily be derived from Eq. (10) as

\[
\Phi_{GTR} = \psi - 2\tan^{-1} \left( \frac{r_1 \sin(\psi)}{1 + r_1 \cos(\psi)} \right).
\]  

(11)

4.2. GTR as a Quasi-optical analogue of Thomas rotation angle

Fig. 5 depicts both the similarities and differences between the expression for the Thomas rotation angle \( \epsilon \) given in Eq. (8) and the expression for the phase response of the GTR, \( \Phi_{GTR} \) in Eq. (11). The side-by-side comparison aims to show that the GTR is clearly far from being a perfect analogue of Thomas rotation angle and, at first glance, an unlikely choice for a potential analogue of Thomas rotation angle. Mathematically speaking, we observe that these two equations shared the same 2\tan^{-1} [ ] expressions.

Fig. 5 also highlights graphically the differences of their corresponding profiles side-by-side. However, with some corrections and important changes in the set-up, this unlikely and unconventional choice (i.e. GTR) for the Thomas angle analogue will prove to be instrumental in modelling the Thomas rotation angle. These two equations (i.e. Eqs. (8) and (11)) can be made mathematical equivalent under the following three conditions.

**Condition 1:** We need to establish an analogy between the two non-linear terms, 2 \( \tan^{-1} \) [ ], in both equations by associating the following parameters namely:

(a) \( r_1 = -V_1V_2 \), the GTR’s reflectance coefficient, \( r_1 \), in Eq. (11) is equated with the product of the two magnitudes of the relativistic velocities \( V_1 \) and \( V_2 \), and
(b) $\psi = \theta$: the single-pass round-trip phase shift $\psi$ of GTR is equated with the “generating angle $\theta$” of the two velocities $V_1$ and $V_2$ in Einstein velocity addition.

**Condition 2:** We need to generate a corresponding linear term $\varphi$, and subtract it from the GTR’s phase response to effective remove the linear term in Eq. (11) to make the two equations mathematical equivalent.

For now, if we assume that these two conditions are fulfilled (detailed discussion is given in Section 5), then the corrected phase response of GTR, $\phi_{GTR}^{\text{C}}(\psi = \theta, r_1 = V_1/V_2)$ can be expressed as

$$\phi_{GTR}^{\text{C}}(\psi = \theta, r_1 = V_1/V_2) = \phi_{GTR} - \psi = -2 \tan^{-1} \left[ \frac{V_1 V_2 \sin(\psi)}{1 + V_1 V_2 \cos(\psi)} \right]$$

which is the same as the Thomas rotation angle $\varepsilon$ found in Eq. (8).

Because of these “corrections”, we will be able to use the GTR as quasi-optical analogue of the Thomas rotation angle.

### 5. Michelson-GT Interferometer (MGTI)

Up to now, we have two issues. First, we need to remove the additional linear phase term $\psi$ found in Eq. (11) to make the correct Thomas angle-vs-GTR analogy. Second, since the Thomas angle is contained in the phase component of the reflected signal from GTR, it is not directly measurable! Therefore, we need to introduce an optical configuration to convert this phase-encoded-Thomas-angle into intensity for direct measurement.

Fig. 6 shows such optical configuration that can resolve simultaneously these two issues. It is known as a Michelson–Gires–Tournois interferometer (MGTI) in the technical literature [22–26]. MGTI is an unbalanced Michelson Interferometer (MI) in which one of its reflecting mirrors is replaced with a GTR.

Operationally speaking, an input beam, $E_{\text{in}}$ from a laser source is split by the beamsplitter (BS: 50:50) into two beams of equal intensities, where one beam $E_1$ travels along $L_1$ toward the GTR, while the other beam, $E_2$ travels along $L_2$ ($=L_1 + \Delta L$) toward a fully reflective mirror $M_1$. The beam $E_1$ is incident normal to GTR and circulates inside the GTR before it is reflected back toward the BS, while $E_2$ will also be reflected back by $M_1$ toward the BS as well. Then, both signals $E_1$ and $E_2$ will recombine at the BS. A portion of the combined signal is transmitted toward the photodetector (PD), and is detected as output signal, $E_{\text{out}}$ expressed as

$$E_{\text{out}}(\psi) = \frac{1}{2} \left[ e^{\imath knL_2} + e^{\imath (knL_1 + \phi_{GTR})} \right]$$

with the corresponding resultant output intensity, $I_{\text{out}} = E_{\text{out}} E_{\text{out}}^*$ given as

$$I_{\text{out}}(\psi) = \frac{1}{2} \left( 1 + \cos \left[ (knL_2 - (knL_1 + \phi_{GTR})) \right] \right)$$

$$= \cos^2 \left[ \frac{k(nL_1 + \Delta L) - (knL_1 + \phi_{GTR})}{2} \right]$$

$$= \cos^2 \left[ \frac{knL_1 - \phi_{GTR}}{2} \right]$$

Fig. 6. Schematic of the Thomas-inspired MGTI where one of the mirrors in a typical Michelson Interferometer (MI) is replaced by a GTR. $M_0$ represents the fully reflecting mirror, $M_1$ is a partially reflecting mirror, BS is the beamsplitter, $M_1$ represents the other reflecting mirror of MI, and PD is the photo detector.
If we set the value of the term \( \kappa n \Delta L = \psi \), and use Thomas angle \( \epsilon \) in Eq. (8), then

\[
I_{\text{mult}}(\psi, r_1) = \cos^2 \left[ \frac{\epsilon(\psi = \theta, r_1 = V_1V_2)}{2} \right].
\]

(15)

Here, we assigned the parameter \( r_1 \) to be associated with \( V_1V_2 \) while the parameter \( \psi \) is equated with the generating angle \( \theta \) as discussed in last portion of Section 3. Clearly, the optical configuration (as expressed by Eq. (15)) converts the corrected phase-encoded Thomas rotation angle \( \epsilon(\psi = \theta, r_1 = V_1V_2) \) into intensity that can be measured directly and readily by PD. The effect of scanning the single-pass phase shift \( \psi \) of the GTR (or the generating angle \( \theta \)) is achieved simply by tuning the centre wavelength of the tunable laser source. Thus, these detected intensities are clearly, readily, and directly measurable with this set-up.

Fig. 7a shows the resultant intensity response as a function of the single-pass phase shift \( \psi \) of the GTR (or the generating angle \( \theta \) in SR) for the case \( r_1 = V_1V_2 \) = 0.1, 0.4, 0.75, and 0.98. We observe that the intensity profile is symmetric with respect to the positive and negative \( \theta \). This is due to the squaring operation in Eq. (14). The actual Thomas rotation angle \( \epsilon \) can be obtained from the detected intensity by performing the reverse operation or a square root operation on the measured intensity from the PD. Physically speaking, this implies that we cannot distinguish the sign of the generating angle from the PD. The effect of scanning the single-pass phase shift \( \psi \) (the Thomas rotation angle \( \theta \)) is achieved simply by tuning the centre wavelength of the tunable laser source. Thus, these detected intensities are clearly, readily, and directly measurable with this set-up.

Fig. 7b captures the essential characteristics of the Thomas rotation angle \( \epsilon \) as discussed in Section 3. When the value of \( r_1 = V_1V_2 \) is very low (less than 0.1), the intensity is nearly flat with a value near unity across the whole range of the generating angle \( \theta \). This supports the case shown in Fig. 3 when the Thomas Angle \( \epsilon(\psi, r_1 = V_1V_2) \) has a small value. Again, the only difference is the sign of the Thomas rotation angle when the generating angle is negative. On the other hand, when the value of \( r_1 = V_1V_2 \) is very high (greater than 0.98) as shown in Fig. 7a, the resultant intensity resembles the standard sinusoidal function with a peak value located at \( \theta = 0 \). This intensity decreases toward zero as the generating angle \( \theta \) increases toward \( +\pi \) or \( -\pi \). It is worth noting that when the value of the generating angle \( \theta \) is nearly equal to \( \pm \pi \), the intensity shoots up rapidly toward the value of unity. The corresponding profile for the Thomas rotation angle for this case is shown in Fig. 7b where the peak value of the Thomas angle never reaches the \( \pi \) value.

6. Conclusion

We presented an optical analogue of the relativistic Thomas rotation angle found in SR using an unconventional starting point—an imperfect optical analogue of the Thomas rotation angle in the form of the phase response from a GTR. Since the GTR’s phase response cannot be measured directly, we incorporated the GTR into a Michelson interferometer (=MGTI) to simultaneously perform two functions namely: (i) to correct the partial analogy, and (2) to convert the phase information into intensity for direct measurement. Then, we investigated both analytically and numerically the Thomas angle’s full behaviour in detail under various parameter conditions.

Overall, the novelty of this work can be described by the scheme we called “imperfect-but-then-corrected-analogy”. We believe that this is the first time that an optical analogue for the thin film-based Thomas rotation angle was constructed using this approach. On the other hand, its associated practical advantages compared with previous approaches are the following: (i) it leads to a simpler overall configuration, (ii) it measures the Thomas rotation angle directly with only a single optical parameter, and (iii) it requires no sophisticated multilayer thin film design.

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